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ON STOCHASTIC PROGRAMMED DESIGN OF STRATEGIES IN A DIFFERENTIAL GAME*

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A differential game /1–17/ is analyzed, in which the strategies form controls on the basis of information on the motion's history. The computation of this game's value is discussed, as also is the construction of optimal strategies on the basis of auxiliary programmed constructions which contain an artificially introduced random element. Thus, a method of stochastic programmed design, proposed in /18,19/ for differential games, is examined here from a certain general viewpoint.

1. Consider the system described by the equation

$$x^{\cdot} = f(t, x, u, v),\ u \in R,\ v \in Q,\ t_0 \leqslant t \leqslant \vartheta \tag{1.1}$$

where t is time, x is the n-dimensional phase vector of the object, u is the r-dimensional control vector, v is the s-dimensional noise vector; the function f is continuous in all arguments and satisfies in x the Lipschitz condition

$$| f(t, x^{(1)}, u, v) - f(t, x^{(2)}, u, v) | \leqslant \lambda\, | x^{(1)} - x^{(2)} |$$

R and Q are compacta; the symbol $|x|$ denotes the Euclidean norm of x. Let the functional

$$\gamma = \gamma(x(t_0[\cdot]\vartheta)) \tag{1.2}$$

be specified, defined on the continuous functions $x(t_0[\cdot]\vartheta) = \{x[t],\ t_0 \leqslant t \leqslant \vartheta$ and continuous in the metric

$$\| x(t_0[\cdot]\vartheta)\| = \max_t | x[t] |$$

The sense of the problem lies in the construction of a control law U which forms a control u by the feedback principle and guarantees the least possible value of γ. As the information element for the current instant $\tau \in [t_0, \vartheta]$ we take the motion history $x(t_0[\cdot]\tau)$, realized up to this instant. Then the problem can be stated as follows. Every function $u(x(t_0[\cdot]\tau), \varepsilon)$ defined for all possible histories $x(t_0[\cdot]\tau)$, $\tau \in [t_0, \vartheta]$, and for sufficiently small values of the precision parameter $\varepsilon > 0$ and satisfying the condition $u(x(t_0[\cdot]\tau), \varepsilon) \in R$ is called a strategy. Suppose that some strategy $u(x(t_0[\cdot]\tau), \varepsilon)$ has been chosen, a history $x(t_0[\cdot]t_*)$ realized, a value $\varepsilon > 0$ and a partitioning $\Delta\{\tau_i\}$, $i = 0, \ldots, m$, for an interval $t_* \leqslant t \leqslant \vartheta$ of future time have been chosen, and let $\tau_0 = t_*, \ldots, \tau_m = \vartheta$. Then the motion $x(t_0[\cdot]\vartheta)$ continuously extending the given history $x(t_0[\cdot]t_*)$ is determined for $t_* \leqslant t \leqslant \vartheta$ as the solution of the stepwise differential equation

$$x^{\cdot}[t] = f(t, x[t], u(x(t_0[\cdot]\tau_i), \varepsilon), v[t]),\ \tau_i \leqslant t \leqslant \tau_{i+1},\ i = 0, \ldots, m-1$$

where the realization of the noise $v(t_*[\cdot]\vartheta) = \{v[t] \in Q,\ t_* \leqslant t < \vartheta\}$ can be any Borel-measurable function, not dependent on our choise. Let the symbol Δ_δ, where $\delta > 0$, denote the partitioning $\Delta\{\tau_i\}$ which satisfies the condition $\tau_{i+1} - \tau_i \leqslant \delta$, $i = 0, \ldots, m-1$. For the chosen strategy $u(\cdot) = u(x(t_0[\cdot]\tau), \varepsilon)$ and for the given initial history $x(t_0[\cdot]t_*)$ the quantity

$$\rho_{u(\cdot)}(x(t_0[\cdot]t_*)) = \overline{\lim_{\varepsilon\to 0}}\, \lim_{\delta\to 0}\, \sup_{\Delta_\delta}\, \sup_{v(t_*[\cdot]\vartheta)} \gamma(x(t_0[\cdot]\vartheta))$$

is called the guaranteed result ρ. The strategy $u^\circ(\cdot)$, which satisfies the condition

$$\rho_{u^\circ(\cdot)}(x(t_0[\cdot]t_*)) = \min_{u(\cdot)} \rho_{u(\cdot)}(x(t_0[\cdot]t_*))$$

for every possible initial history $x(t_0[\cdot]t_*)$ is said to be optimal.

A function $v(x(t_0[\cdot]\tau), u, \varepsilon$ defined for all possible histories $x(t_0[\cdot]\tau)$, for sufficiently small $\varepsilon > 0$ and for all $u \in R$, satisfying the condition $v(x(t_0[\cdot]\tau), u, \varepsilon) \in Q$ and being Borel-measurable in u for each fixed $x(t_0[\cdot]\tau)$ and ε, is called a counter-strategy. Suppose that

*Prikl.Matem.Mekhan.,46,No.6,885-892,1982 some counterstrategy $v(x(t_0[\cdot]\tau), u, \varepsilon)$ has been chosen, a history $x(t_0[\cdot]t_*)$ realized, a value $\varepsilon > 0$ and a partitioning $\Delta\{\tau_i\}$ for the interval $[t_*, \vartheta]$ have been chosen. Then the motion $x(t_0[\cdot]\vartheta)$ continuously extending $x(t_0[\cdot]t_*)$ is determined for $t_* \leqslant t \leqslant \vartheta$ as the solution of the step-wise differential equation

$$\dot{x}[t] = f(t, x[t], u[t], v(x(t_0[\cdot]\tau_i), u[t], \varepsilon)),$$
$$\tau_i \leqslant t \leqslant \tau_{i+1}, \quad i = 0, \ldots, m-1$$

where the realization $u(t_*[\cdot]\vartheta) = \{u[t] \in R, \ t_* \leqslant t < \vartheta\}$ can be any Borel-measurable function. For the chosen counterstrategy $v(\cdot) = v(x(t_0[\cdot]\vartheta), u, \varepsilon)$ and for the given initial history $x(t_0[\cdot]t_*)$ the guaranteed result ρ is determined by the equality

$$\rho_{v(\cdot)}(x(t_0[\cdot]t_*)) = \varliminf_{\varepsilon\to 0} \lim_{\delta\to 0} \inf_{\Delta_\delta} \inf_{u(t_*[\cdot]\vartheta)} \gamma(x(t_0[\cdot]\vartheta))$$

The counterstrategy $v^\circ(\cdot)$ which satisfies the condition

$$\rho_{v^\circ(\cdot)}(x(t_0[\cdot]t_*)) = \max_{v(\cdot)}\rho_{v(\cdot)}(x(t_0[\cdot]t_*))$$

for every possible initial history $x(t_0[\cdot]t_*)$ is said to be optimal.

The problems of optimal strategy $u^\circ(\cdot)$ and optimal counterstrategy $v^\circ(\cdot)$ constitute a differential game. We say that this game has a value ρ° and a saddle point $\{u^\circ(\cdot), v^\circ(\cdot)\}$ if optimal $u^\circ(\cdot)$ and $v^\circ(\cdot)$ exist and the equality

$$\rho_{u^\circ(\cdot)}(x(t_0[\cdot]t_*)) = \rho_{v^\circ(\cdot)}(x(t_0[\cdot]t_*)) = \rho^\circ(x(t_0[\cdot]t_*))$$

is valid for every possible initial history $x(t_0[\cdot]t_*)$.

Theorem 1.1. The differential game being analyzed has a value and a saddle point. The theorem can be proved in a well-known way (see /4,18,20/, for example). In this connection the optimal $u^\circ(\cdot)$ and $v^\circ(\cdot)$ can be constructed from the game's value ρ° according to the conditions

$$\max_v \langle(x[\tau] - y_*[\tau])\cdot f(\tau, x[\tau], u^\circ, v)\rangle =$$
$$\min_u \max_v \langle(x[\tau] - y_*[\tau])\cdot f(\tau, x[\tau], u, v)\rangle$$
$$\langle(x[\tau] - y^*[\tau])\cdot f(\tau, x[\tau], u, v^0)\rangle =$$
$$\min_v \langle(x[\tau] - y^*[\tau])\cdot f(\tau, x[\tau], u, v)\rangle$$

Here the symbol $\langle a\cdot b\rangle$ denotes the scalar product of vectors a and b, $y_*[\tau]$ and $y^*[\tau]$ are the values at instant τ for the accompanying histories $y_*(t_0[\cdot]\tau)$ and $y^*(t_0[\cdot]\tau)$, which are determined from the conditions

$$\rho^\circ(y_*(t_0[\cdot]\tau)) = \min \rho^\circ(y(t_0[\cdot]\tau)) \tag{1.3}$$
$$\rho^\circ(y^*(t_0[\cdot]\tau)) = \max \rho^\circ(y(t_0[\cdot]\tau)) \tag{1.4}$$

under the constraint $\| x(t_0[\cdot]\tau) - y(t_0(\cdot)\tau)\| \leqslant \varepsilon \exp 2\lambda(\tau - t_0)$. In the general case the considered proof of Theroem 1.1 is not constructive. Therefore, the effective computation of ρ° and the construction of $u^\circ(\cdot)$ and $v^\circ(\cdot)$ remains an unsolved problem.

2. We consider an estimate of the quantity $\rho^\circ(x(t_0[\cdot]t_*))$, relying on a certain auxiliary stochastic programmed construction. Suppose that some history $x(t_0[\cdot]t_*)$ has been fixed. For the interval $[t_*, \vartheta]$ we select a certain partitioning $\Delta\{t_i\}$, $i = 1, \ldots, k+1$, $t_1 = t_*, \ldots, t_{k+1} = \vartheta$. With this partitioning we connect the following probability space $\{\Omega, F, P\}$. In this space an elementary event ω is any set $\omega = \{z^{(1)}, \ldots, z^{(k)}, s^{(1)}, \ldots, s^{(k)}\}$, where $z^{(j)}$ and $s^{(j)}$ are n-dimensional vectors, $|s^{(j)}| \leqslant 1$, $|z^{(j)}| \leqslant K$, where K is some sufficiently large number, F is a Borel σ-algebra on Ω, and the probability measure P is generated by aggregate-independent uniform distributions of random vectors $z^{(j)}$ and $s^{(j)}$ in the corresponding balls $|z| \leqslant K$ and $|s| \leqslant 1$. Thus, we assume that with instant t_j there is connected a pair $\{z^{(j)}, s^{(j)}\}$ of random variables distributed uniformly for $|z| \leqslant K$ and $|s| \leqslant 1$; all the variables $z^{(j)}, s^{(j)}$ are independent in aggregate.

Random functions Borel-measurable in all their arguments, nonanticipatory (in t) relative to $\xi[t_*, t] = \{z^{(1)}, \ldots, z^{(i)}, s^{(1)}, \ldots, s^{(i)}\}$, $t_i \leqslant t < t_{i+1}$ /21/, are called nonanticipatory stochastic programs $v(t, u, \omega)$ and $u(t, \omega)$. Therefore, these functions satisfy the equalities

$$v(t, u, \omega) = v(t, u, z^{(1)}, \ldots, z^{(i)}, s^{(1)}, \ldots, s^{(i)})$$
$$u(t, \omega) = u(t, z^{(1)}, \ldots, z^{(i)}, s^{(1)}, \ldots, s^{(i)})$$

for $t_i \leqslant t < t_{i+1}$, $i = 1, \ldots, k$, almost surely in ω. The history $x(t_0[\cdot]t_*)$, the partitioning $\Delta\{t_i\}$ and some pair $\{u(t,\omega), v(t, u, \omega)\}$ of programs determine a random motion $w(t_0[\cdot]\vartheta, \omega)$, which continuously extends this history for $t_* \leqslant t \leqslant \vartheta$, as the solution of the stochastic differential equation

$$w^{\cdot}[t] = f(t, w[t,\omega], u(t,\omega), v(t, u(t,\omega), \omega)) \tag{2.1}$$

By $v_\Delta(\cdot)$ and $u_\Delta(\cdot)$ we denote programs corresponding to partition Δ. The quantity

$$\rho^*(x(t_0[\cdot]\ t_*)) = \sup_\Delta \sup_{v_\Delta(\cdot)} \inf_{\Delta(\cdot)} \operatorname{ess\,max}_\omega \gamma(w(t_0[\cdot]\vartheta, \omega))$$

where $\operatorname{ess\,max}_\omega\gamma$ is computed for the random variable γ on the space $\{\Omega, F, P\}$, is called the programmed maximin ρ^*.

Theorem 2.1. The equality

$$\rho^\circ(x(t_0[\cdot]\ t_*)) = \rho^*(x(t_0[\cdot]\ t_*)) \tag{2.2}$$

is valid for any history $x(t_0[\cdot]\ t_*)$. The theorem's proof is a consequence of the following lemmas.

Lemma 2.1. For any history $x(t_0[\cdot]\ t_*)$, partitioning Δ, program $v_\Delta(\cdot)$ and number $\beta > \rho^\circ(x(t_0[\cdot]\ t_*))$ there exists a program $u_\Delta(\cdot)$ which with probability one ensures the inequality

$$\gamma(w(t_0[\cdot]\vartheta, \omega)) < \beta$$

for the corresponding motion $w(t_0[\cdot]\vartheta, \omega)$ from (2.1).

The lemma's validity follows from the well-known property of u-stability of function ρ°: for any history $w(t_0[\cdot]\ t_i, \omega)$, number $\alpha > 0$ and admissible function $v(t_i[\cdot]t_{i+1}, u, \omega)$ we can find an admissible function $u(t_i[\cdot]\ t_{i+1}, \omega)$ (all for a fixed value of ω) such that the inequality

$$\rho^\circ(w(t_0[\cdot]\ t_{i+1}, \omega)) \leqslant \rho^\circ(w(t_0[\cdot]\ t_i, \omega)) + \alpha(t_{i+1} - t_i) \tag{2.3}$$

is fulfilled for the corresponding motion $w(t_0[\cdot]\ t_{i+1}, \omega)$. Since here the function $u(t_i[\cdot]\ t_{i+1}, \omega)$ can be taken to be Borel-measurable in t and ω (in t and $\{z^{(1)}, \ldots, z^{(i)}, s^{(1)}, \ldots, s^{(i)}\}$), the lemma can be proved directly by induction on the basis of inequalities (2.3), starting from $\rho^\circ(x(t_0[\cdot]\ t_*)) = \rho^\circ(w(t_0[\cdot]\ t_*))$ and ending at $\rho^\circ(w(t_0[\cdot]\vartheta, \omega)) = \gamma(w(t_0[\cdot]\vartheta, \omega))$.

Lemma 2.2. For any history $x(t_0[\cdot]\ t_*)$ and number $\beta < \rho^\circ(x(t_0[\cdot]|t_*))$ there exists a partitioning Δ and a program $v_\Delta(\cdot)$ such that for every program $u_\Delta(\cdot)$ the inequality

$$P(\gamma(w(t_0[\cdot]\vartheta, \omega)) > \beta) > \zeta > 0 \tag{2.4}$$

is ensured for the corresponding motion $w(t_0[\cdot]\ \vartheta, \omega)$ from (2.1), where the symbol $P(A)$ denotes the probability of event A.

Indeed, we shall assume the step $\delta = \max_i(t_{i+1} - t_i)$ of partitioning $\Delta\{t_i\}$ to be sufficiently small (an estimate of the suitable smallness of δ will be indicated below). We determine the program $v(t, u, \omega) = v(t_i, u, z^{(i)}, s^{(i)})$, $t_i \leqslant t < t_{i+1}$, $i = 1, \ldots, k$, so as to fulfil the condition

$$\langle s^{(i)} \cdot f(t_i, z^{(i)}, u, v(t_i, u, z^{(i)}, s^{(i)}))\rangle = \min_v \langle s^{(i)} \cdot f(t_i, z^{(i)}, u, v)\rangle \tag{2.5}$$

From the measurable selection lemma /22/ follows the possibility of constructing such a function $v(t, u, \omega)$, measurable in the arguments $z^{(i)}, s^{(i)}$ and u for $t_i \leqslant t < t_{i+1}$, $i = 1, \ldots, k$. We consider the random motion $w(t_0[\cdot]\vartheta, \omega) = \{w[t, \omega],\ t_0 \leqslant t \leqslant \vartheta, \omega \in \Omega\}$, extending the history $x(t_0[\cdot]\ t_*)$ and generated by the constructed program $v_\Delta(\cdot)$ of (2.5) and some program $u_\Delta(\cdot)$. Let $y^*[t_i, \omega]$ be the value at instant t_i for the accompanying history $y^*(t_0[\cdot]\ t_i, \omega)$, which is determined from condition (1.4), where $\tau = t_i$, $x(t_0[\cdot]\ t_i) = w(t_0[\cdot]\ t_i, \omega)$ and $\varepsilon > 0$ is some sufficiently small number. We select some value $\alpha > 0$. Because of the independence of the random variables $z^{(j)}, s^{(j)}$, $j = 1, \ldots, k$ we can assert the validity of the inequality

$$P(|z^{(i)}(\omega) - w[t_i, \omega]| \leqslant \alpha, |s^{(i)}(\omega) - (w[t_i, \omega] - y^*[t_i, \omega])| \leqslant \alpha \mid w(t_0[\cdot]\ t_i, \omega)) \geqslant \eta(\alpha) > 0 \tag{2.6}$$

Here the symbol $P(A|\xi)$ denotes the conditional probability of event A with respect to the random variable (function) ξ. Because of the continuity of function $f(t, x, u, v)$ and the v-stability of function ρ° /4,20/, by well-known arguments /4,18,20/ we can now infer from conditions (2.5) and (2.6) that we can find a certain (random) history $y(t_0[\cdot]\ t_{i+1}, \omega)$ which extends the history $y^*(t_0[\cdot]\ t_i, \omega)$ and is such that the inequality

$$\begin{aligned}
&P(\| w(t_i[\cdot]\ t_{i+1}, \omega) - y(t_i[\cdot]\ t_{i+1}, \omega)\|^2 \leqslant \\
&\quad (w[t_i, \omega] - y^*[t_i, \omega])^2(1 + 2\lambda(t_{i+1} - t_i) + \zeta(\delta, \alpha)(t_{i+1} - t_i)), \\
&\rho^\circ(y(t_0[\cdot]\ t_{i+1}, \omega)) \geqslant \rho^\circ(y^*(t_0[\cdot]\ t_i, \omega)) - \zeta(\delta, \alpha)(t_{i+1} - t_i) \\
&\mid w(t_0[\cdot]\ t_i, \omega)) \geqslant \eta(\alpha) > 0
\end{aligned}$$

where

$$\lim \zeta(\alpha, \delta) = 0, \alpha \to 0, \delta \to 0$$

is valid. Hence by induction we can prove that for any not preselected values $\zeta^* > 0$ and $\varepsilon > 0$ we can find arbitrarily small values $\delta > 0$ and $\alpha > 0$ such that for the motion $w(t_0[\cdot]\vartheta, \omega)$ from (2.1) being examined we can find a history $y(t_0[\cdot]\vartheta, \omega)$ for which the inequality

$$P(\|w(t_0[\cdot]\vartheta,\omega) - y(t_0[\cdot]\vartheta,\omega)\| \leqslant \varepsilon \exp 2\lambda(\vartheta - t_0),$$
$$\rho^\circ(y(t_0[\cdot]\vartheta,\omega)) \geqslant \rho^\circ(y^*(t_0[\cdot]t_*,\ \omega)) - \zeta^*,$$
$$\|x(t_0[\cdot]t_*) - y^*(t_0[\cdot]t_*,\ \omega)\| \leqslant \varepsilon \exp 2\lambda(t_* - t_0)) \geqslant \eta^k(\alpha) \tag{2.7}$$

is valid. As a consequence of the continuity of ρ° and γ, from (2.7) it follows that for any value $\beta < \rho^\circ(x(t_0[\cdot]t_*))$, for the choice of sufficiently small $\varepsilon > 0$ and $\zeta^* > 0$ and for sufficiently small $\delta > 0$ and $\alpha > 0$ the program $v_\Delta(\cdot)$ constructed for (2.4) ensures inequality (2.4) for every program $u_\Delta(\cdot)$. This proves Lemma 2.2.

3. In the general case the computation of the game's value ρ° in terms of the programmed maximin ρ^* on the basis of equality (2.2) is scarcely constructive. However, in certain cases this equality is useful for estimating ρ° and for constructing $u^\circ(\cdot)$ and $v^\circ(\cdot)$. Here we consider in detail the case when the equation of motion (1.1) is strictly linear, i.e.

$$x^{\cdot} = A(t)x + f(t, u, v),\ u \in R,\ v \in Q \tag{3.1}$$

where $A(t)$ is a continuous matrix-valued function. In this case we can construct the probability space $\{\Omega, F, P\}$ on the basis of only the random variables $s^{(i)}$, assuming $\omega = \{s^{(1)}, \ldots, s^{(k)}\}$, since in the case of (3.1) the variables $z^{(i)}$ in condition (2.5) do not really play any role. (In general, in each actual case the space $\{\Omega, F, P\}$ can be selected as this or that depending on the selection of realizations of some random function $\xi(t_0[\cdot]\vartheta, \omega)$ or other as ω, whose nature corresponds to the problem at hand).

On the given probability space $\{\Omega, F, P\}$ we now choose a certain normed linear space $L^{(2)}([t_0, \vartheta], \Omega)$ of random functions $w(t_0[\cdot]\vartheta, \cdot) = \{w(t, \omega), t_0 \leqslant t \leqslant \vartheta, \omega \in \Omega\}$, continaing the random functions $w(t,\omega)$ with continuous (almost sure) realizations $w(t_0[\cdot]\vartheta, \omega)$. We assume that the given functional γ of (1.2) can in some way or other be extended onto the realizations $w(t_0[\cdot]\vartheta, \omega)$ of the elements from $L^{(2)}$ so that we can speak of the random variable $\gamma[\omega] = \gamma(w(t_0[\cdot]\vartheta, \omega))$. Here we assume the fulfillment of the following condition. Let $W_\beta^{(2)}$ be the set of elements $w(\cdot) = w(t_0[\cdot]\vartheta, \cdot)$ from $L^{(2)}$, which satisfy the condition

$$\text{ess max}_\omega\, \gamma(w(t_0[\cdot]\vartheta, \omega)) \leqslant \beta \tag{3.2}$$

and let the inequality

$$P(\gamma(w(t_0[\cdot]\vartheta, \omega)) \geqslant \beta + \varepsilon) \geqslant \zeta \tag{3.3}$$

be valid for some values $\varepsilon > 0$ and $\zeta > 0$ for some element $w(\cdot) \in L^{(2)}$. Then the inequality

$$\varphi \geqslant \eta(\beta, \varepsilon, \zeta) > 0 \tag{3.4}$$

is valid for the distance φ in $L^{(2)}$ from the element $w(\cdot)$ to set $W_\beta^{(2)}$.

In what follows we take for definiteness, for example, that the norm $\|w(\cdot)\|_{(2)}$ in $L^{(2)}$ is specified by the inequality

$$\|w(\cdot)\|_{(2)} = \Big(M\int_{t_0}^{\vartheta} |w(t,\omega)|^2 \mu(dt)\Big)^{1/2} = \Big(\int_\Omega\int_{t_0}^{\vartheta} |w(t,\omega)|^2 \mu(dt)P(d\omega)\Big)^{1/2}$$

and that the norm in the adjoint space $L_*^{(2)}([t_0, \vartheta], \Omega)$ of random functions $l(\cdot) = l(t_0[\cdot]\vartheta, \omega) = \{l(t, \omega), t_0 \leqslant t \leqslant \vartheta, \omega \in \Omega\}$ is specified by the equality

$$\|l(\cdot)\|_{(2)}^* = \Big(M\int_{t_0}^{\vartheta} |l(t,\omega)|^2 \mu(dt)\Big)^{1/2}$$

where $\mu(dt)$ is some Borel measure on the interval $[t_0, \vartheta]$ and the symbol M denotes the mean (mathematical expectation).

For a given initial history $x(t_0[\cdot]t_*)$ we select some number $\beta < \rho^0(x(t_0[\cdot]t_*))$. Let some function $w^{(2)}(t_0[\cdot]\vartheta, \cdot) \in L^{(2)}$ satisfy condition (3.2). We choose a program $v_\Delta(\cdot)$ constructed in accordance with condition (2.5), under the assumption that the base space $\{\Omega, F, P\}$ corresponds to a partitioning Δ of interval $t_* \leqslant t \leqslant \vartheta$, having a sufficiently small step δ. Scanning all possible nonanticipatory stochastic programs $u_\Delta(\cdot)$, in space $L^{(2)}$ we obtain an attainability domain G composed of all possible random motions $w(t_0[\cdot]\vartheta, \cdot)$ from (2.1), extending the history $x(t_0[\cdot]t_*)$. If the step δ of partitioning $\Delta\{t_i\}$ is sufficiently small, then by Lemma 2.2 the domain G cannot contain the function $w^{(2)}(t_0[\cdot]\vartheta,\cdot)$, because for a sufficiently small step δ, for all the motions $w(t_0(\cdot)\vartheta,\cdot)$ from (2.1) being examined, the condition (2.4) is fulfilled for some $\zeta > 0$ and $\varepsilon > 0$. Furthermore, inequality (3.4) is fulfilled then for the distance φ in $L^{(2)}$ from any motion $w(t_0[\cdot]\vartheta, \cdot)$ to the element $w^{(2)}(t_0[\cdot]\vartheta, \cdot)$.

By arguments well known in the theory of strictly linear controlled systems we can show that the closure in $L^{(2)}$ of domain G coincides with the closed convex hull $W^{(1)}$ in $L^{(2)}$ of this domain G . Therefore, the inequality

$$\varphi^* \geqslant \eta(\beta, \varepsilon, \zeta) > 0 \tag{3.5}$$

also will be fulfilled for the distance φ^* in $L^{(2)}$ from the element $w^{(2)}(t_0[\cdot]\vartheta, \cdot)$ to the closed convex hull of domain G . Hence, once again on the basis of well-known arguments from the theory of strictly linear systems /4/, which rely on the theorem on the separation of convex sets in $L^{(2)}$, we deduce the inequality (the prime denote transposition)

$$\begin{aligned}
&\varphi(x(t_0[\cdot]t_*), \Delta, w^{(2)}(t_0[\cdot]\vartheta, \cdot)) = \\
&\sup_{l(\cdot)}\Big[M\int_{t_*}^{t_*} \langle l(t,\omega)\cdot x[t]\rangle \mu(dt) + \\
&M\int_{t_*}^{\vartheta} \langle l(t,\omega)\cdot X(\vartheta,t)\, x[t_*]\rangle \mu(dt) + \\
&M\int_{t_*}^{\vartheta} \min_u \langle M_\tau\Big[\int^{\vartheta} X'(\vartheta,t)\, l(t,\omega)\mu(dt)\Big]\cdot f(\tau,u,v(\tau,u,\omega))\rangle d\tau - \\
&M\int_{t_*}^{\vartheta} \langle l(t,\omega)\cdot w^{(2)}(t,\omega)\rangle \mu(dt)\Big] \geqslant \eta(\beta,\varepsilon,\zeta), \ \| l(\cdot)\|_{(2)}^* \leqslant 1
\end{aligned} \tag{3.6}$$

because the quantity φ in (3.6) is also the distance φ^* . Here $X(t, t^*)$ is the fundamental matrix of solutions of the homogeneous equation $x^{\cdot} = A(t)x$ and the symbol $M_\tau[\xi(\omega)]$ denotes the conditional mean

$$M_\tau[\xi(\omega)] = M[\xi(\omega)\,|\, s^{(1)}(\omega), \ldots, s^{(i)}(\omega)], \quad t_i \leqslant \tau < t_{i+1}, \qquad i = 1, \ldots, k$$

Conversely, if some function $w^{(2)}(t_0[\cdot]\vartheta, \cdot) \in W^{(1)}$, then $\varphi(x(t_0[\cdot]t_*), \Delta, w^{(2)}(t_0[\cdot]\vartheta, \cdot)) = 0.$ From these relations, with due regard to Lemmas 2.1 and 2.2, we conclude that the programmed maximin $\rho^*(x(t_0[\cdot]t_*))$, and the game's value $\rho^\circ(x(t_0[\cdot]t_*))$ equal to it, is the upper bound of those numbers β for which the inequality

$$\sup_\Delta \inf_{w^{(2)}(\cdot)} \varphi(x(t_0[\cdot]t_*), \Delta, w^{(2)}(t_0[\cdot]\vartheta, \cdot)) > 0 \tag{3.7}$$

is valid when $w^{(2)}(t_0[\cdot]\vartheta, \cdot) \in W_\beta^{(2)}$.

If the sets $W_\beta^{(2)}$ are convex for every β , then the quantity $\rho^* = \rho^\circ$ is the upper bound of those values of β for which the inequality

$$\sup_\Delta \varphi(x, (t_0[\cdot]t_*), \Delta, W_\beta^{(2)}) > 0 \tag{3.8}$$

is valid, where the quantity φ differs from the quantity φ from (3.6) only by the last summand which for the φ from (3.8) has the form

$$\xi = -\sup_{w^{(2)}(\cdot)} M\int_{t_*}^{\vartheta} \langle l(t,\omega)\cdot w^{(2)}(t,\omega)\rangle \mu(dt)$$

when $w^{(2)}(t_0[\cdot]\vartheta, \cdot) \in W_\beta^{(2)}$. Thus, the problem is reduced to computing the quantity φ for (3.7) or for (3.8). The problem of computing φ is a mathematical programming problem on the maximum of a functional concave in $l(\cdot)$, under the constraint $\| l(\cdot)\|_{(2)}^* \leqslant 1$. Such a problem has no principal difficulties; however, the practical computations often prove to be too laborious.

Above we have considered the case when the original differential game can be formalized in the classes (strategies-counterstrategies). In completely the same way we can prove a theorem, similar to Theorem 2.1, also in the case when the game has been formalized in classes of mixed strategies. The only difference here is that in the programmed stochastic construction the programs of the functions $u(t,\omega)$ and $v(t,u,\omega)$ are replaced by the programs of the measures $\mu(t,\omega)$ on R and $\nu(t,\omega)$ on Q.

In conclusion, let us compare the quantity φ involved in the programmed stochastic construction in the case of convex sets $W_\beta^{(2)}$ with corresponding quantities in the analogous problems treated in /18,19/. The difference is determined by the fact that here the computation is based always on one and the same universal program $v_\Delta(\cdot)$ from (2.5), whereas in /18, 19/ the program $v(\cdot)$ is based each time as the extremal program corresponding to the problem's properties on the maximum over $l(\cdot)$. The transition to the universal program $v_\Delta(\cdot)$ improves certain qualities of the problem on the computation of φ : the quantity to be maximized becomes a function concave in $l(\cdot)$, etc. At the same time the quantity φ itself loses certain useful properties. Thus, generally speaking, the quantity φ loses the property of u-stability; only the quantity ρ^* is left with it. This obtains because, in general, it is not assumed that the nature of space $L^{(2)}$ corresponds to the nature of functional γ and to the nature of the estimate of the deviations of the random variable $\gamma[\omega]$ from the quantity $\mathrm{ess}\max_\omega \gamma[\omega]$. Thus, for the sake of definiteness, above we chose the Hilbert space $L^{(2)}$. This determines the convenient adjoint space $L_*^{(2)}$. But in many cases the property of u-stability can be returned to the quantity φ also in the case of the universal program $v_\Delta(\cdot)$ if the metric in $L^{(2)}$ is selected in accordance with the nature of functional γ and with the estimate of the deviation of $\gamma[\omega]$ from the quantity $\mathrm{ess}\max_\omega\gamma[\omega]$.

For example, if the functional $\gamma(x(t_0[\cdot]\vartheta)) = \nu(x[\vartheta])$ and the quantity $\gamma(x)$ has the sense of some norm $\|x\|$ in space $\{x\}$, then as the norm $\|w(\cdot)\|_{(2)} = \|w(\vartheta,\cdot)\|_{(2)}$ we can choose the quantity $\|w(\vartheta,\cdot)\|_{(2)} = \mathrm{ess}\max_\omega |w(\vartheta,\omega)\|$. Then the quantity φ acquires the appropriate property of u-stability. But the computation of φ is complicated by the fact that the adjoint space now turns out to be the space of additive functions $\lambda(A)$ of the subsets $A \in \Omega$. However, this is not necessarily too complicated a matter, since often the problem later is again reduced to suitable functions $l(\omega)$ of the points $\omega \in \Omega$. If the quantity γ does not have the sense of a norm, then again we can strive to return the u-stability property to the quantity φ by defining the latter not in terms of distance in $L^{(2)}$ but on the basis of estimates which are determined by the functions γ_* (or the functionals γ_*) adjoint in due manner to the quantity γ.

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